,

Intermediate Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

Solutions and investigations

1 February 2023

These solutions augment the shorter solutions also available online. The shorter solutions in many cases omit details. The solutions given here are full solutions, as explained below. In some cases alternative solutions are given. There are also many additional problems for further investigation. We welcome comments on these solutions and the additional problems. Please send them to challenges@ukmt.org.uk.

The Intermediate Mathematical Challenge (IMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that sometimes you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the IMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with each step explained (or, occasionally, left as an exercise). We therefore hope that these solutions can be used as a model for the type of written solution that is expected when a complete solution to a mathematical problem is required (for example, in the Intermediate Mathematical Olympiad and similar competitions).

Enquiries about the Intermediate Mathematical Challenge should be sent to:

IMC, challenges@ukmt.org.uk

www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
D	A	B	D	B	E	B	E	B	D	B	E	A	C	A	A	C	B	E	B	A	E	D	C	D

1. Which of these numbers is neither a multiple of 3, nor a multiple of 4?

A 16 B 21 C 28 D 34 E 45

Solution **D**

16 is a multiple of 4, 21 is a multiple of 3, 28 is a multiple of 4, and 45 is a multiple of 3.

However 34 is neither a multiple of 3 nor a multiple of 4.

For investigation

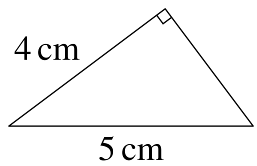
1.1 For each of the following integers n find the number of integers in the range from 1 to n inclusive which are neither a multiple of 3, nor a multiple of 4.

(a) 4, (b) 5, (c) 6, (d) 48. (e) 100, (f) 2023.

1.2 Determine for which positive integers n there are exactly $\frac{1}{2}n$ integers in the range from 1 to n inclusive which are neither a multiple of 3, nor a multiple of 4.

2. What is the area of this triangle?

A $6\,\text{cm}^2$ B $7.5\,\text{cm}^2$ C $8\,\text{cm}^2$
D $10\,\text{cm}^2$ E $12\,\text{cm}^2$



Solution **A**

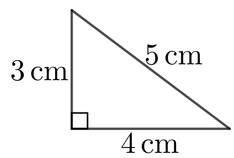
Let the length of the third side of the right-angled triangle shown in the question be x cm.

By Pythagoras' Theorem, applied to this triangle, $4^2 + x^2 = 5^2$. It follows that

$$x^2 = 5^2 - 4^2 = 25 - 16 = 9.$$

Hence $x = 3$.

If we take the side with length 4 cm as the base of the triangle, the height of the triangle is the length of the perpendicular side with length 3 cm. Therefore, from the formula



$$\text{area of a triangle} = \tfrac{1}{2}(\text{base} \times \text{height}),$$

it follows that the area of the triangle is $\frac{1}{2}(4 \times 3)\,\text{cm}^2 = 6\,\text{cm}^2$.

For investigation

2.1 PQR is a right-angled triangle whose shortest side has length 5 cm and whose area is $30\,\text{cm}^2$. What is the length of the hypotenuse of the triangle PQR?

2.2 STU is a right-angled triangle whose side lengths, in centimetres, are all integers. One side of the triangle has length 7 cm. Find the area and perimeter of the triangle.

3. What is the value of $1 - (2 - (3 - 4 - (5 - 6)))$?

A -2 B -1 C 0 D 1 E 2

SOLUTION **B**

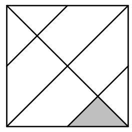
We have

$$\begin{aligned} 1 - (2 - (3 - 4 - (5 - 6))) &= 1 - (2 - (3 - 4 - (-1))) \\ &= 1 - (2 - (3 - 4 + 1)) \\ &= 1 - (2 - 0) \\ &= 1 - 2 \\ &= -1. \end{aligned}$$

FOR INVESTIGATION

3.1 What is the value of $1 - (2 - (3 - (4 - (5 - (6 - 7)))))$?

4. The diagram shows a square, its two diagonals and two line segments, each of which connects two midpoints of sides of the square.

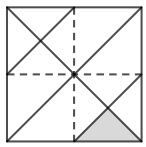


What fraction of the area of the square is shaded?

A $\frac{1}{8}$ B $\frac{1}{10}$ C $\frac{1}{12}$ D $\frac{1}{16}$ E $\frac{1}{24}$

SOLUTION **D**

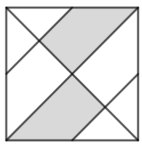
As may be seen from the diagram on the right, the shaded area is $\frac{1}{4}$ of $\frac{1}{4}$ of the square.



Therefore the fraction of the square that is shaded is $\frac{1}{4} \times \frac{1}{4}$, that is, $\frac{1}{16}$.

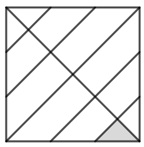
FOR INVESTIGATION

4.1 The diagram shows, as in the question, a square, its two diagonals and two line segments joining midpoints of sides.



What fraction of the area of the square is shaded?

4.2 (a) The diagram shows a square, its two diagonals, and four line segments joining points which divide the sides of the square into three equal parts.



What fraction of the area of the square is shaded?

(b) What fraction of the area of the square would be shaded when there are n line segments joining points which divide the sides of the square into $n - 1$ equal parts?

5. We know that $1 + 2 + 3 + 4 = 10$. It is also true that $1^3 + 2^3 + 3^3 + 4^3 = 10^n$ for some integer n.
What is this integer?

A 1 B 2 C 3 D 4 E 5

Solution **B**

We have

$$\begin{aligned} 1^3 + 2^3 + 3^3 + 4^3 &= 1 + 8 + 27 + 64 \\ &= 100 \\ &= 10^2. \end{aligned}$$

Therefore $n = 2$.

For investigation

5.1 Note that the solution above shows that

$$1^3 + 2^3 + 3^3 + 4^3 = (1 + 2 + 3 + 4)^2.$$

Check that also

(a) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 = (1 + 2 + 3 + 4 + 5)^2$, and

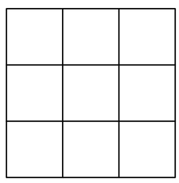
(b) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 = (1 + 2 + 3 + 4 + 5 + 6)^2$.

5.2 In fact it is true that for every positive integer n,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.$$

Find a proof of this (that is, work out a proof yourself, or look in a book or on the web, or ask your teacher).

6. To draw a 3 by 3 square grid you need 8 straight lines, as shown.



How many straight lines do you need to draw an n by n square grid?

A $n + 5$ B $3n - 1$ C $n^2 - 1$ D $4(n - 1)$ E $2(n + 1)$

Solution **E**

To draw an n by n grid, you need to draw $n + 1$ horizontal lines and $n + 1$ vertical lines.

Therefore you need to draw $2(n + 1)$ lines altogether.

For investigation

6.1 For which value of n are there are 61 more cells in an n by n square grid than the number of lines needed to draw the grid?

7. What is 0.015% of 60 million?

A 900 B 9000 C 90 000 D 900 000 E 9 000 000

SOLUTION **B**

0.015% is $\frac{15}{1000}$%, that is, $\frac{15}{1000} \times \frac{1}{100} = \frac{15}{100\,000}$.

60 million is 60 000 000.

Therefore 0.015% of 60 million $= \frac{15}{100\,000} \times 60\,000\,000 = 15 \times \frac{60\,000\,000}{100\,000} = 15 \times 600 = 9000$.

FOR INVESTIGATION

7.1 What is 0.0025% of 120 million?

7.2 The population of the world is estimated to be eight billion. The population of Leeds is estimated to be eight hundred thousand.

Approximately, what percentage of the population of the world lives in Leeds?

7.3 Approximately 0.8% of the population of the world lives in the UK. Approximately 14% of the population of the UK lives in London.

Approximately, what percentage of the population of the world lives in London?

8. $\sqrt{\sqrt{x}} = 3$. What is the value of x?

A $\sqrt{\sqrt{3}}$ B $\sqrt{3}$ C 9 D 12 E 81

SOLUTION **E**

Since

$$\sqrt{\sqrt{x}} = 3,$$

by squaring both sides of this equation, it follows that

$$\sqrt{x} = 3^2 = 9.$$

Therefore, squaring again,

$$x = 9^2 = 81.$$

FOR INVESTIGATION

8.1 $\sqrt{\sqrt{x}} = 7$. What is the value of x?

8.2 $\sqrt{x} = x - 12$. What is the value of x?

8.3 Find the solutions of the equation $\sqrt{x} = 2\sqrt{\sqrt{x}}$.

9. Merryn wrote down the numbers 2, 0, 2, 3 and one further number.
What is the median of her five numbers?

A 0 B 2 C 2.5 D 3
E more information required

Solution **B**

Method 1

From the wording of the question we assume that the answer does not depend on the value of Merryn's fifth number. So we suppose that Merryn's fifth number is 0.

Then in order of magnitude her numbers are 0,0,2,2,3. Their median is 2.

Note: This quick method is all right in the context of the IMC. But if you were required to give a full mathematical explanation, you would need to give a detailed argument as in Method 2.

Method 2

The median of Merryn's five numbers is the number which is in the middle when the numbers are listed in order of magnitude. In other words, it is the third number in the list.

Let the one other number that Merryn writes down be x.

When Merryn's five numbers are listed in order of magnitude, the order in which they will appear depends on the size of x in relation to the other numbers 0, 2 and 3 in the list.

There are therefore four cases to consider, as shown in the following table.

x	order of the five numbers	median
$x \leq 0$	$x, 0, \mathbf{2}, 2, 3$	2
$0 < x \leq 2$	$0, x, \mathbf{2}, 2, 3$	2
$2 < x \leq 3$	$0, 2, \mathbf{2}, x, 3$	2
$3 < x$	$0, 2, \mathbf{2}, 3, x$	2

Therefore, whatever the value of Merryn's fifth number, the median is 2.

For investigation

9.1 Merryn wrote down the numbers $1, 2, 2, 3$,$3, 3, 4, 4, 4, 4$ and one further number.

What is the median of her eleven numbers?

9.2 Merryn wrote down the numbers $1, 2, 2, 3, 3$ and one further integer.

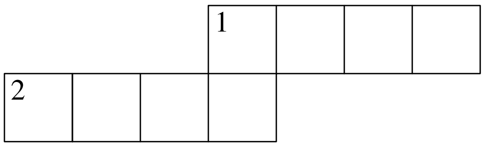
What are the possible values of the median of her six numbers?

9.3 Merryn wrote down the numbers $1, 2, 2, 3, 3$ and one further number which might not be an integer.

What are the possible values of the median of her six numbers?

10.

Across	**Down**
1. A power of 5	1. A power of 6
2. A power of 4	



Eight of the digits from 0 to 9 inclusive are used to fill the cells of the crossnumber. What is the sum of the two digits which are not used?

A 12 B 13 C 14 D 15 E 16

Solution **D**

The first few powers of 5 are $5^1 = 5$, $5^2 = 25$, $5^3 = 125$, $5^4 = 625$, $5^5 = 3125$ and $5^6 = 15\,625$.

Therefore 3125 is the only 4-digit power of 5. It follows that 1 Across is 3125.

Similarly, the only 2-digit power of 6 is $6^2 = 36$. Therefore 1 Down is 36.

The first few powers of 4 are $4^1 = 4$, $4^2 = 16$, $4^3 = 64$, $4^4 = 256$, $4^5 = 1024$, $4^6 = 4096$ and $4^7 = 16\,384$. Thus, the only 4-digit powers of 4 are 1024 and 4096.

To fit in with 1 Down, the units digit of 2 Across needs to be 6. Therefore, 2 Across is 4096.

Hence the completed crossnumber is as below.

			[1] 3	1	2	5
[2] 4	0	9	6			

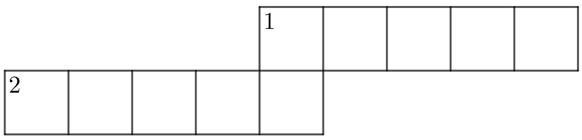
Between them 3125, 36 and 4096 use all the digits other than 7 and 8.

Therefore the sum of the two digits that are not used is $7 + 8 = 15$.

For investigation

10.1 (a) Complete this crossnumber.

Across	**Down**
1. A power of 11	1. A square
2. A power of 2	



(b) How many different digits are used in the solution to this crossnumber?

(c) Invent your own crossnumber with the same grid, and which uses fewer different digits in its solution.

11. Jill was given a large jar of jam. She gave one sixth of the jam to Jan. Jill then gave one thirteenth of the remaining jam to Jas. Jill was left with 1 kg of jam.
What was the weight, in kg, of the jam in Jill's jar at the start?

A 1.2 B 1.3 C 1.4 D 1.6 E 1.9

Solution **B**

Let x be the weight, in kg, of the jam in Jill's jar at the start.

After Jill has given one sixth of the jam to Jan, she is left with five sixths of the jam. That is, Jill now has $\frac{5}{6}x$ kg of jam.

Jill now gives one thirteenth of this jam to Jas. She is left with twelve thirteenths of the jam she had before giving jam to Jas. Therefore she now has $\frac{12}{13}\left(\frac{5}{6}x\right)$ kg of jam.

Since we are told that Jill is left with 1 kg of jam, it follows that

$$\frac{12}{13}\left(\frac{5}{6}x\right) = 1.$$

Therefore

$$x = \frac{13}{12} \times \frac{6}{5} = \frac{13}{10} = 1.3.$$

So the weight, in kg, of the jam in Jill's jar at the start was 1.3.

For investigation

11.1 The next day Jill was given another large jar of jam. She gave one thirteenth of the jam to Jan, and then gave one sixth of the remaining jam to Jas. Jill was again left with 1 kg of jam in the jar.

What was the weight, in kg, of Jill's jam at the start of the day.

11.2 The next day Jill was given another large jar containing 1.6 kg of jam. She gave a certain fraction of this jam to Jan. Jill then gave one eighth of the remaining jam to Jas. Jill was again left with 1 kg of jam in the jar,

What fraction of her original amount of jam did Jill give to Jan?

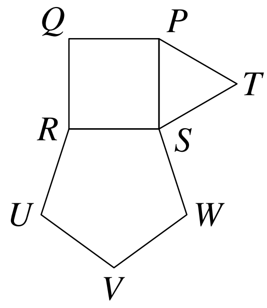
11.3 The next day Jill was given yet another large jar containing 1.44 kg of jam. She gave a certain fraction of this jam to Jan. Jill then gave the same fraction of the remaining jam to Jas. Jill was again left with 1 kg of jam in the jar,

What fraction of her original amount of jam did Jill give to Jan?

12. In the diagram, $PQRS$ is a square, PST is an equilateral triangle and $SRUVW$ is a regular pentagon.

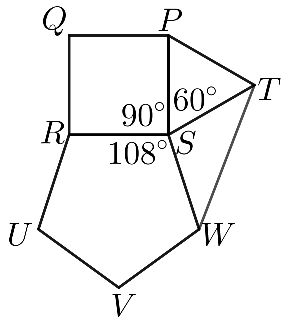
What is the size of angle WTS?

A $35°$ B $36°$ C $37°$ D $38°$ E $39°$



SOLUTION **E**

The angles in the equilateral triangle PST are each $60°$, the angles in the square $PQRS$ are each $90°$ and the angles in the regular pentagon $SRUVW$ are each $108°$.



Because the angles at the point S add up to $360°$,

$$\angle WST = 360° - (60° + 90° + 108°) = 102°.$$

The sum of the angles in the triangle WTS is $180°$. Therefore

$$\angle WTS + \angle TWS = 180° - \angle WST = 180° - 102° = 78°.$$

Since $ST = PS = RS = SW$, the triangle WTS is isosceles.

Therefore $\angle WTS = \angle TWS$.

Hence $\angle WTS = \frac{1}{2}(78°) = 39°$.

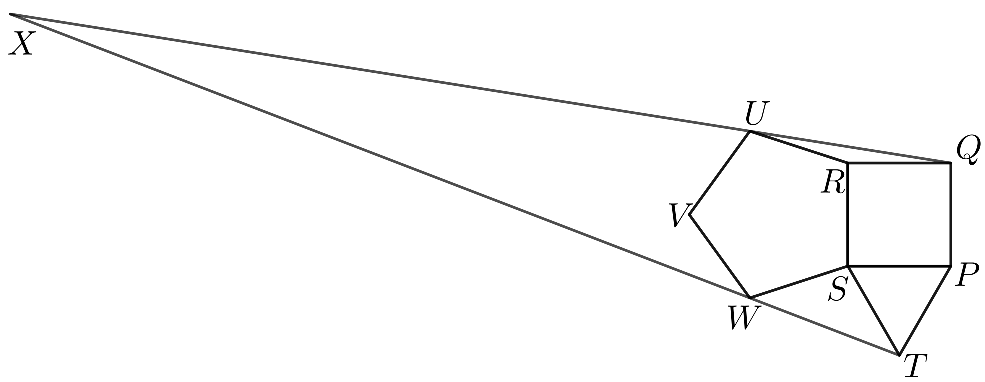
FOR INVESTIGATION

12.1 Show that the interior angles of a regular pentagon are each $108°$.

12.2 You are told that the interior angles of a particular regular polygon are each $160°$.

How many edges does this regular polygon have?

12.3 In the diagram of the question, let X be the point where the lines QU and TW meet when extended.



What is the size of the angle QXT?

13. The mean of p and q is 13; the mean of q and r is 16; the mean of r and p is 7. What is the mean of p, q and r?

A 12 B 13 C 14 D 15 E 16

Solution **A**

Since the mean of p and q is 13, $\frac{p+q}{2} = 13.$ (1)

Since the mean of q and r is 16, $\frac{q+r}{2} = 16.$ (2)

Since the mean of r and p is 7, $\frac{r+p}{2} = 7.$ (3)

By adding equations (1), (2) and (3), we obtain $p + q + r = 36$.

Therefore, the mean of p, q and r is given by

$$\frac{p+q+r}{3} = \frac{36}{3} = 12.$$

For investigation

13.1 The mean of s and t is 7. The mean of t and u is 12. The mean of s, t and u is 9. What is the mean of s and u?

14. A regular octagon $PQRSTUVW$ has sides of length 2 cm. When I shade the rectangles $PQTU$ and $RSVW$, four small triangles inside the octagon remain unshaded. What is the total area, in cm^2, of these four triangles?

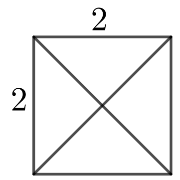
A 1 B 2 C 4 D 6 E 8

Solution **C**

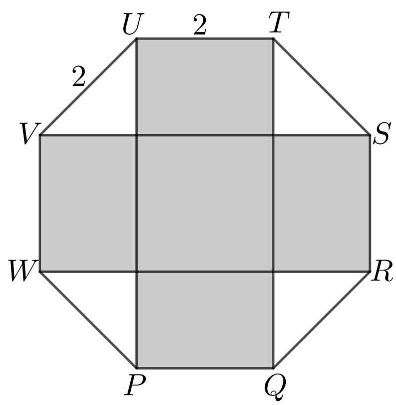
It may be seen that each of the four unshaded triangles is a right-angled isosceles triangle.

The hypotenuse of each of these triangles has length 2 cm.

These four triangles fit together to make a square with side length 2 cm.



It follows that the total area, in cm^2, of these four triangles is 4.



For investigation

14.1 What is the shaded area in the diagram on the right above?

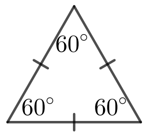
15. How many of the following polygons could exist?

A triangle with all three sides the same length, but three different interior angles.
A quadrilateral with all four sides the same length, but four different interior angles.
A pentagon with all five sides the same length, but five different interior angles.

A only the pentagon
B only the quadrilateral
C the quadrilateral and the pentagon
D all three
E none of them

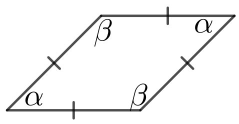
Solution **A**

A triangle with all three sides the same length is an equilateral triangle. Therefore all its interior angles are equal to $60°$.



Hence a triangle with all three sides the same length, but with three different interior angles does not exist.

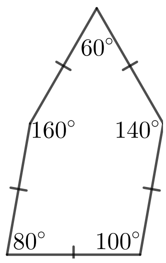
A quadrilateral with all four sides the same length is a *rhombus*. Therefore it has two pairs of opposite interior angles which are equal, as shown in the diagram on the right. (You are asked to prove this in Problem 15.1.)



Hence a quadrilateral with all four sides the same length, but with four different interior angles does not exist.

However, pentagons with all five sides the same length, but with five different interior angles do exist.

An example of a pentagon with these properties is shown in the diagram on the right.



It follows that the option that is correct is A.

For investigation

15.1 Prove that in a rhombus the opposite interior angles are equal.

15.2 Find another example of a pentagon in which all the sides are the same length and the interior angles, in degrees, are given by five different integers.

15.3 Explore the possibilities for the interior angles of a pentagon all of whose sides have the same length.

Health Warning: There is no easy straightforward answer to this problem.

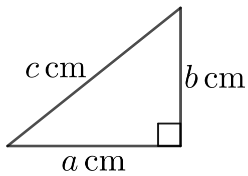
15.4 Can you construct a pentagon in which all the interior angles are equal, but all the side lengths are different?

16. The sum of the the lengths of the three sides of a right-angled triangle is 16 cm. The sum of the squares of the lengths of the three sides of the triangle is 98 cm^2.
What is the area, in cm^2, of the triangle?

A 8 B 10 C 12 D 14 E 16

SOLUTION **A**

We let the lengths of the sides of the triangle be a cm, b cm and c cm, with c cm being the length of the hypotenuse.



From the formula area $= \frac{1}{2}(\text{base} \times \text{height})$, it follows that the area of the triangle is $\frac{1}{2}ab\,\text{cm}^2$.

From the information given in the question, we have

$$a + b + c = 16 \qquad (1) \qquad \text{and} \qquad a^2 + b^2 + c^2 = 98. \qquad (2)$$

Also, by Pythagoras' theorem,

$$a^2 + b^2 = c^2. \qquad (3)$$

By (2) and (3), $2c^2 = 2(a^2 + b^2) = 98$. Hence

$$c^2 = a^2 + b^2 = 49. \qquad (4)$$

Therefore

$$c = 7. \qquad (5)$$

From (1) and (5)

$$a + b = 9. \qquad (6)$$

We can now deduce from (4) and (6) that

$$\begin{aligned}
\tfrac{1}{2}ab &= \tfrac{1}{4}(2ab) \\
&= \tfrac{1}{4}\big((a^2 + 2ab + b^2) - (a^2 + b^2)\big) \\
&= \tfrac{1}{4}\big((a + b)^2 - (a^2 + b^2)\big) \\
&= \tfrac{1}{4}(9^2 - 49) \\
&= \tfrac{1}{4}(81 - 49) \\
&= \tfrac{1}{4}(32) \\
&= 8.
\end{aligned}$$

Therefore the area, in cm^2, of the triangle is 8.

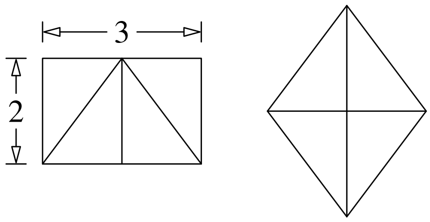
FOR INVESTIGATION

16.1 Note that in the above solution we were able to find the value of ab directly from the equations $a + b = 9$ and $a^2 + b^2 = 49$ without the need to first find the separate values of a and b.

Use the equations $a + b = 9$ and $a^2 + b^2 = 49$ to find the values of both (i) $a^3 + b^3$ and (ii) $a^4 + b^4$.

16.2 Use the equations $a + b = 9$ and $a^2 + b^2 = 49$ to find the values of a and b.

17. A 3 by 2 rectangle is split into four congruent right-angled triangles, as shown in the left-hand diagram.
Those four triangles are rearranged to form a rhombus, as shown in the right-hand diagram.
What is the ratio of the perimeter of the rectangle to the perimeter of the rhombus?



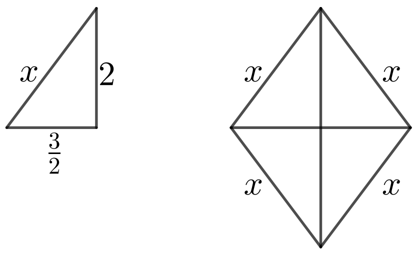
A 3 : 2 B 2 : 1 C 1 : 1 D 1 : 2 E 2 : 3

Solution **C**

The rectangle has two sides of length 3 and two sides of length 2. Therefore the perimeter of the rectangle is given by

$$2 \times 3 + 2 \times 2 = 10.$$

We let x be the length of the hypotenuse of the congruent right-angled triangles. These triangles are obtained from the standard 3, 4, 5 right-angled triangle by scaling by the factor $\frac{1}{2}$. Therefore $x = \frac{5}{2}$.



It follows that the length of perimeter of the rhombus is given by

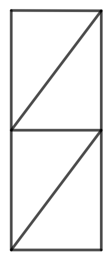
$$4 \times x = 4 \times \tfrac{5}{2} = 10.$$

Therefore the ratio of the perimeter of the rectangle to the perimeter of the rhombus is $10 : 10 = 1 : 1$.

For investigation

17.1 The diagram on the right shows another shape made from the four congruent right-angled triangles of this question.

What is the perimeter of this shape?



17.2 Find a way to rearrange the four congruent right-angled triangles of this question to make a connected shape with a perimeter of length 14.

17.3 What is the maximum perimeter of a connected shape formed using the four right-angled triangles of this question?

17.4 Check directly that $x = \frac{5}{2}$ by verifying that

$$(\tfrac{5}{2})^2 = 2^2 + (\tfrac{3}{2})^2.$$

18. How many squares are exactly four greater than a prime?

A 0 B 1 C 2 D 3 E 4

Solution **B**

Commentary

A good way to begin thinking about this question is to list the first few primes, and the integers which are four greater than these primes. This gives

$$p: \quad 2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ \dots$$

$$p+4: \quad 6,\ 7,\ 9,\ 11,\ 15,\ 17,\ 21,\ 23,\ \dots$$

You will notice that there is just one square, 9, in the second row. This may lead you to guess that the correct answer is B. A more satisfactory mathematical approach is to tackle the general case using algebra.

Let p be a prime with the property that $p + 4$ is a square. Then there is a positive integer n such that

$$p + 4 = n^2.$$

It follows that

$$p = n^2 - 4 = (n - 2)(n + 2).$$

Since p is a prime the factorization of p as $(n - 2) \times (n + 2)$, must be $1 \times p$.

Therefore $n - 2 = 1$. Hence $n = 3$ and so $n^2 = 9$.

This proves that 9 is the only square that is 4 more than a prime. Hence there is just one such prime.

For investigation

18.1 The above solution shows that there is just one prime number which may be expressed in the form $n^2 - 4$, where n is an integer.

In this problem you are asked to investigate primes of the form $n^2 + 4$.

(a) Explain why, if n is even, the number $n^2 + 4$ is not prime.

(b) Find the smallest odd integer n for which $n^2 + 4$ is a prime.

(c) Find the smallest odd integer n for which $n^2 + 4$ is *not* a prime.

(d) Show that there are infinitely many odd integers n for which $n^2 + 4$ is not a prime.

Note: This question does not settle the issue as to whether there are infinitely many primes of the form $n^2 + 4$. At the time of writing, this is an open question whose answer is not known.

19. What is the positive difference between the numerator and the denominator when the expression shown is written as a single fraction in its simplest form?

$$\frac{n}{n+1-\dfrac{n+2}{n+3}}$$

A $2n+2$ B $n+2$ C n D 2 E 1

Solution **E**

We have

$$\begin{aligned}\frac{n}{n+1-\dfrac{n+2}{n+3}} &= \frac{n}{\left(\dfrac{(n+1)(n+3)-(n+2)}{n+3}\right)}\\ &= \frac{n}{\left(\dfrac{(n^2+4n+3)-(n+2)}{n+3}\right)}\\ &= \frac{n}{\left(\dfrac{n^2+3n+1}{n+3}\right)}\\ &= \frac{n(n+3)}{n^2+3n+1}.\end{aligned}$$

Since n^2+3n+1 does not factorize into linear factors with integer coefficients, the last expression cannot be simplified further.

For all integers n,

$$n^2+3n+1 > n(n+3).$$

Therefore the positive difference between the numerator and the denominator is given by

$$(n^2+3n+1)-n(n+3) = n^2+3n+1-n^2-3n = 1.$$

For investigation

19.1 Simplify the expression

$$\frac{\dfrac{1}{n+1}-\dfrac{1}{n+2}}{\dfrac{1}{n+2}-\dfrac{1}{n+3}}.$$

19.2 Show that for every integer n,

$$\frac{n}{\dfrac{n+1}{n+2}-\dfrac{n+2}{n+3}}$$

is also an integer.

19.3 How can we be sure that n^2+3n+1 does not factor into linear factors with integer coefficients?

20. I roll two standard six-sided fair dice. At least one of the scores obtained on the dice is 3. What is the probability that both of the scores on the dice are 3?

A $\frac{1}{12}$ B $\frac{1}{11}$ C $\frac{1}{6}$ D $\frac{1}{3}$ E $\frac{1}{4}$

SOLUTION **B**

COMMENTARY

You need to think very carefully about probability problems of this kind, as it is very easy to fall into traps.

For example, why is the following argument *WRONG*?

> The score on one of the two dice is 3. The probability that the score on the other dice is 3 is $\frac{1}{6}$. Therefore the probability that both of the scores are 3 is $\frac{1}{6}$.

The correct solution below shows that the answer $\frac{1}{6}$ is wrong. Can you explain where the argument above goes astray?

In the table on the right we have set out all the possible outcomes when the two dice are rolled. In each cell, the first number gives the score on the first dice, and the second number gives the score on the second dice.

$1,1$	$1,2$	$1,3$	$1,4$	$1,5$	$1,6$
$2,1$	$2,2$	$2,3$	$2,4$	$2,5$	$2,6$
$3,1$	$3,2$	$3,3$	$3,4$	$3,5$	$3,6$
$4,1$	$4,2$	$4,3$	$4,4$	$4,5$	$4,6$
$5,1$	$5,2$	$5,3$	$5,4$	$5,5$	$5,6$
$6,1$	$6,2$	$6,3$	$6,4$	$6,5$	$6,6$

Because both dice are fair, the 36 outcomes in this table are all equally likely.

We see that in 11 of these outcomes at least one of the scores is 3. (These are the outcomes whose cells are shaded.)

In just one of these 11 cases the scores on both the dice is 3.

Therefore the required probability is $\frac{1}{11}$.

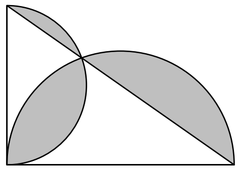
FOR INVESTIGATION

20.1 I roll two standard six-sided fair dice. The score on one of the dice is 3. What is the probability that the score on the other dice is 4?

20.2 I roll three standard six-sided fair dice. The score on at least one of the dice is 3. What is the probability that the score on all three dice is 3?

20.3 I roll three standard six-sided fair dice. The score at least one of the dice is 3. What is the probability that the score on at least two of the dice is 3?

21. A semicircle of radius 3 units is drawn on one edge of a right-angled triangle, and a semicircle of radius 4 units is drawn on another edge. The semicircles intersect on the hypotenuse of the triangle, as shown.
What is the shaded area, in square units?



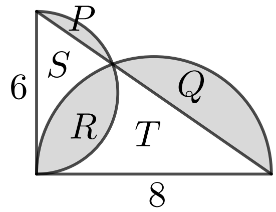
A $\frac{25\pi}{2} - 24$ B 12 C $\frac{25\pi}{2} - 6$ D $25\pi - 24$ E 24

Solution **A**

Commentary

It may be seen that the area covered by the two semicircles is the area of the triangle plus the shaded area. Hence the shaded area is the difference between the sum of the areas of the semicircles and the area of the triangle. We make this argument more explicit using algebra, as follows.

We label the areas of the regions in the figure as shown the diagram on the right. We use the formula $\frac{1}{2}\pi r^2$ for the area of a semicircle with radius r.



The area of the semicircle with radius 3 is $\frac{1}{2}(\pi 3^2)$, that is, $\frac{9\pi}{2}$.

Hence $$P + R + S = \frac{9\pi}{2}. \qquad (1)$$

The area of the semicircle with radius 4 is $\frac{1}{2}(\pi 4^2)$, that is, 8π.

Hence $$Q + R + T = 8\pi. \qquad (2)$$

The right-angled triangle has perpendicular sides with lengths equal to the diameters of the semicircles. These lengths are 6 and 8, as shown. Therefore the area of this triangle is $\frac{1}{2}(6 \times 8)$, that is, 24.

Hence $$R + S + T = 24. \qquad (3)$$

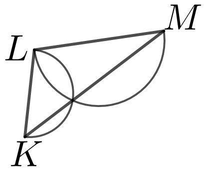
Now, $$P + Q + R = (P + R + S) + (Q + R + T) - (R + S + T).$$

Therefore, by (1), (2) and (3), the shaded area, in square units, is given by

$$\frac{9\pi}{2} + 8\pi - 24 = \frac{25\pi}{2} - 24.$$

For investigation

21.1 KLM is a triangle with semicircles drawn on the edges KL and LM as shown. Prove that if these semicircles meet, the point where they meet is on the edge KM.



[*Hint*: What do you know about the angle in a semicircle?]

22. The numbers x and y satisfy both of the equations

$$\boxed{23x + 977y = 2023 \qquad \text{and} \qquad 977x + 23y = 2977.}$$

What is the value of $x^2 - y^2$?

A 1 B 2 C 3 D 4 E 5

SOLUTION **E**

COMMENTARY

It should be apparent that here the standard methods for solving a pair of linear equations would lead to some horrendous arithmetic. For example, using the first equation to substitute for y in the second equation would give $977x + 23\left(\frac{2023 - 23x}{977}\right) = 2977$. This isn't a nice equation to have to solve without a calculator. Therefore you should look for something better.

We are given that

$$23x + 977y = 2023 \qquad (1)$$

and

$$977x + 23y = 2977. \qquad (2)$$

By adding equations (1) and (2) we obtain

$$1000x + 1000y = 5000.$$

It follows that

$$x + y = 5. \qquad (3)$$

By subtracting equation (1) from equation (2) we obtain

$$954x - 954y = 954.$$

It follows that

$$x - y = 1. \qquad (4)$$

From equations (3) and (4) it follows that

$$x^2 - y^2 = (x - y)(x + y) = 1 \times 5 = 5.$$

FOR INVESTIGATION

22.1 Find the values of x and y that satisfy the two equations given in this question, and check that they satisfy the equation $x^2 - y^2 = 5$.

22.2 The numbers x and y satisfy both of the equations

$$\boxed{19x + 81y = 1138 \qquad \text{and} \qquad 81x + 19y = 1262.}$$

What is the value of $x^2 - y^2$?

23. It is possible to choose, in two different ways, six different integers from 1 to 9 inclusive such that their product is a square. Let the two squares so obtained be p^2 and q^2, where p and q are both positive.
What is the value of $p + q$?

A 72 B 84 C 96 D 108 E 120

Solution **D**

In the prime factorization of a square each prime occurs to an even power. Conversely, if in the prime factorization of an integer each prime occurs to an even power, the integer is a square.

The prime factorization of the product of all the integers from 1 to 9 inclusive is given by

$$\begin{aligned} 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 &= 1 \times 2 \times 3 \times 2^2 \times 5 \times (2 \times 3) \times 7 \times 2^3 \times 3^2 \\ &= 2^7 \times 3^4 \times 5 \times 7. \end{aligned}$$

We therefore see that to obtain a square from the product of six of the integers from 1 to 9 we need to exclude from the product 5 and 7 and a third integer which is an odd power of 2. Therefore, we need to exclude either 5, 7 and 2, or 5, 7 and 8.

In this way we obtain the following two squares

$$\begin{aligned} 1 \times 3 \times 4 \times 6 \times 8 \times 9 &= 1 \times 3 \times 2^2 \times (2 \times 3) \times 2^3 \times 3^2 \\ &= 2^6 \times 3^4 \\ &= (2^3 \times 3^2)^2 \\ &= (8 \times 9)^2 \\ &= 72^2 \end{aligned}$$

and

$$\begin{aligned} 1 \times 2 \times 3 \times 4 \times 6 \times 9 &= 1 \times 2 \times 3 \times 2^2 \times (2 \times 3) \times 3^2 \\ &= 2^4 \times 3^4 \\ &= (2^2 \times 3^2)^2 \\ &= (4 \times 9)^2 \\ &= 36^2. \end{aligned}$$

It follows that, with $p, q > 0$, p and q are 72 and 36, or 36 and 72. In either case

$$p + q = 108.$$

For investigation

23.1 How many different squares are there which are the product of six different integers from 1 to 10 inclusive?

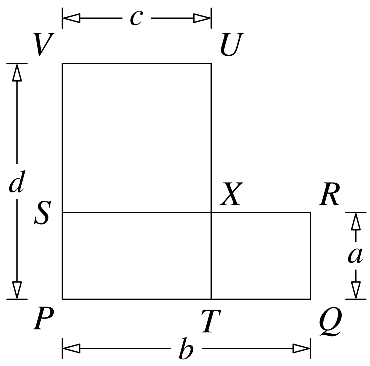
23.2 (a) How many squares are there which are factors of the product

$$1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10\,?$$

(b) How many of these squares are the product of different integers all of which are in the range from 1 to 10 inclusive?

24. A rectangle $PQRS$ has side-lengths a and b, with $a < b$. The rectangle $PTUV$ has side-lengths c and d, with $c < d$. Also, $a < d$ and $c < b$, as shown. The sides RS and TU cross at X.

Which of these conditions guarantees that Q, X and V lie on a straight line?



A $\frac{a}{b} + \frac{c}{d} = 1$ B $\frac{a}{c} + \frac{b}{d} = 1$

C $\frac{a}{d} + \frac{c}{b} = 1$ D $\frac{a}{c} + \frac{d}{b} = 1$

E $\frac{c}{a} + \frac{b}{d} = 1$

Solution **C**

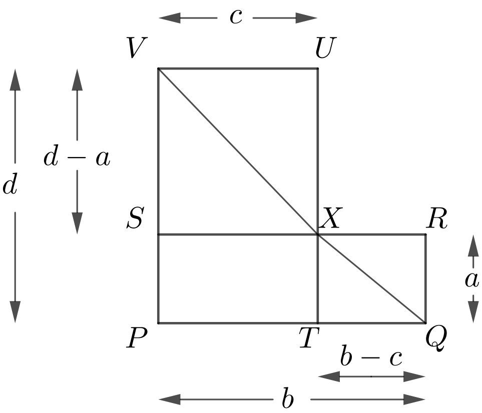
If the line VX has the same slope as the line XQ then the points Q, X and V lie on a straight line.

Therefore the condition $\frac{SV}{SX} = \frac{TX}{TQ}$ guarantees that Q, X and V lie on a straight line.

From the diagram we see that $\frac{SV}{SX} = \frac{TX}{TQ} \Leftrightarrow \frac{d-a}{c} = \frac{a}{b-c}$.

Now

$$\begin{aligned}\frac{d-a}{c} = \frac{a}{b-c} &\Leftrightarrow (d-a)(b-c) = ac\\ &\Leftrightarrow bd - cd - ab + ac = ac\\ &\Leftrightarrow bd = ab + cd\\ &\Leftrightarrow \frac{ab+cd}{bd} = 1\\ &\Leftrightarrow \frac{a}{d} + \frac{c}{b} = 1.\end{aligned}$$



[Note that, since b and d are lengths, $bd \neq 0$.]

Therefore the condition that guarantees that Q, X and V lie on a straight line is $\frac{a}{d} + \frac{c}{b} = 1$.

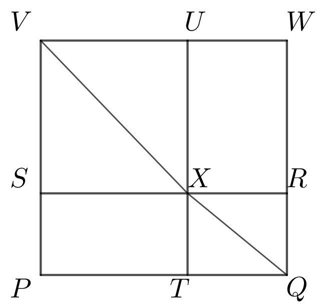
For investigation

24.1 Let W be the point where VU extended meets QR extended so that $PQWV$ is a rectangle. Show that

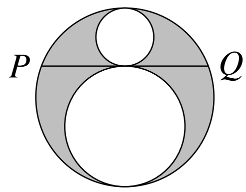
Q, X and V lie on a straight line

is equivalent to

the rectangles $PTXS$ and $XRWU$ have the same area.



25. The diagram shows two unshaded circles which touch each other and also touch a larger circle. Chord PQ of the larger circle is a tangent to both unshaded circles. The length of PQ is 6 units.
What is the area, in square units, of the shaded region?



A 3π B $\frac{7\pi}{2}$ C 4π D $\frac{9\pi}{2}$ E 5π

Solution **D**

Commentary

Note that the question does not specify the diameter of the larger circle, except that, as PQ has length 6 units, the diameter of the circle must be at least 6 units.

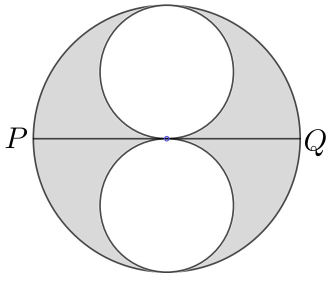
Therefore, in the context of the IMC, it is safe to assume that the answer is independent of the size of the larger circle. We can therefore choose a length for the diameter so as to make the problem as simple as possible.

We adopt this approach in Method 1 where we make the convenient assumption that the diameter of the larger circle is 6 units. It follows that PQ is a diameter of the larger circle. The calculation is then very easy.

In Method 2 we do not make this assumption. Instead we give a general argument which makes no assumption about the size of the larger circle. In this case the calculation is more complicated , but it shows that the answer is independent of the size of the larger circle.

Method 1

We assume that PQ is a diameter of the larger circle.



It follows that the larger circle has radius 3 units, and the smaller circles each have radius $\frac{3}{2}$ units.

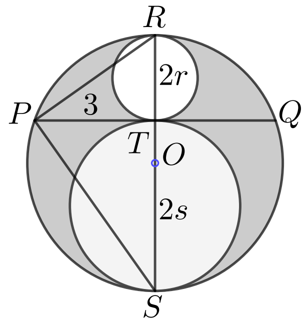
The shaded area is the area of the larger circle less the area of the two smaller circles.

Using the formula πr^2 for the area of a circle with radius r, it follows that the shaded area is given, in square units, by

$$\begin{aligned}\pi(3^2) - 2 \times \pi\left(\frac{3}{2}\right)^2 &= 9\pi - 2 \times \frac{9\pi}{4}\\ &= 9\pi - \frac{9\pi}{2}\\ &= \frac{9\pi}{2}.\end{aligned}$$

Method 2

We let O be the centre of the largest circle, and R, S, T be the points where the circles touch, as shown in the diagram.



Then RT and TS are diameters that are perpendicular to the tangent PQ. It follows that RTS is a straight line which is a diameter of the largest circle.

We let r be the radius of the smallest circle, s be the radius of the other unshaded circle, and t be the radius of the largest circle.

The area that is shaded is found by subtracting the areas of the unshaded circles from the area of the largest circle.

Therefore the shaded area is $\pi t^2 - \pi r^2 - \pi s^2$, that is, $\pi(t^2 - r^2 - s^2)$.

Now $RS = RT + TS$ and therefore $2t = 2r + 2s$. Hence $t = r + s$. It follows that the shaded area is given by

$$\begin{aligned}\pi((r+s)^2 - r^2 - s^2) &= \pi((r^2 + 2rs + s^2) - r^2 - s^2)\\ &= 2\pi rs. \qquad (1)\end{aligned}$$

Because RS is a diameter, $\angle RPS = 90°$. It follows that the triangle RTP is similar to the triangle PTS (you are asked to check this in Problem 25.1).

Therefore $\dfrac{RT}{PT} = \dfrac{PT}{TS}$. Hence $\dfrac{2r}{3} = \dfrac{3}{2s}$. It follows that $2rs = \dfrac{9}{2}$. Therefore, by (1), the shaded area, in square units, is $\dfrac{9\pi}{2}$.

(For an alternative way to find the value of $2rs$ see Problem 25.2.)

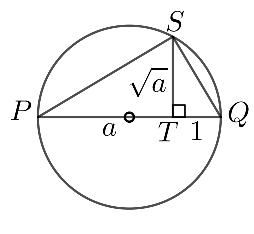
For investigation

25.1 Prove that the triangle RTP is similar to the triangle PTS.

25.2 Apply Pythagoras' Theorem to the right-angled triangle TOP to find the value of $2rs$.

25.3 In the diagram on the right, PQ is a diameter of the circle, $PT = a$ and $TQ = 1$.

Prove that $ST = \sqrt{a}$.



Note: This shows that if there is a straight edge and compass construction to construct a line with length a, then there is also a straight edge and compass construction to construct a line with length $\sqrt{a}$.